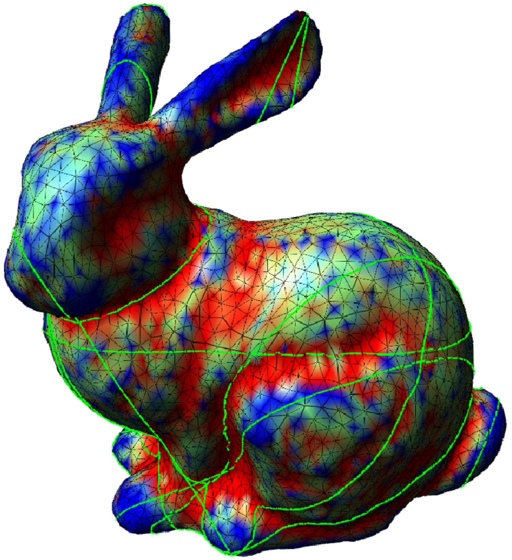
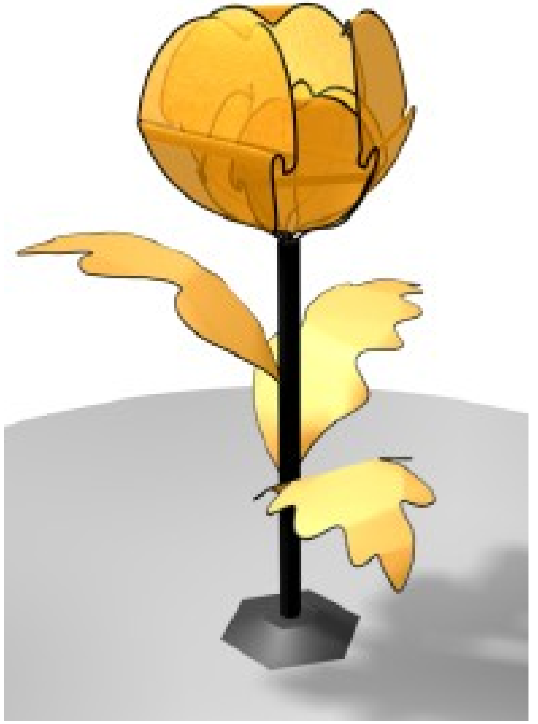
Creating and processing 3D geometry



Marie-Paule Cani
Marie-Paule.Cani@imag.fr

Cédric Gérot
Cedric.Gerot@gipsa-lab.inpg.fr

Franck Hétroy
Franck.Hetroy@imag.fr



http://evasion.imag.fr/Membres/Franck.Hetroy/Teaching/Geo3D/

Planning (provisional)

Part I – Geometry representations

- Lecture 1 – Oct 9th – FH
  - Introduction to the lectures; point sets, meshes, discrete geometry.
- Lecture 2 – Oct 16th – MPC
  - Parametric curves and surfaces; subdivision surfaces.
- Lecture 3 – Oct 23rd - MPC
  - Implicit surfaces.

Planning (provisional)

Part II – Geometry processing

- **Lecture 4 – Nov 6th – FH**
  - **Discrete differential geometry; mesh smoothing and simplification (paper presentations).**
- Lecture 5 – Nov 13th - CG + FH
  - Mesh parameterization; point set filtering and simplification.
- Lecture 6 – Nov 20th - FH (1h30)
  - Surface reconstruction.

Planning (provisional)

Part III – Interactive modeling

- Lecture 6 – Nov 20th – MPC (1h30)
  - Interactive modeling techniques.
- Lecture 7 – Dec 04th - MPC
  - Deformations; virtual sculpting.
- Lecture 8 – Dec 11th - MPC
  - Sketching; paper presentations.

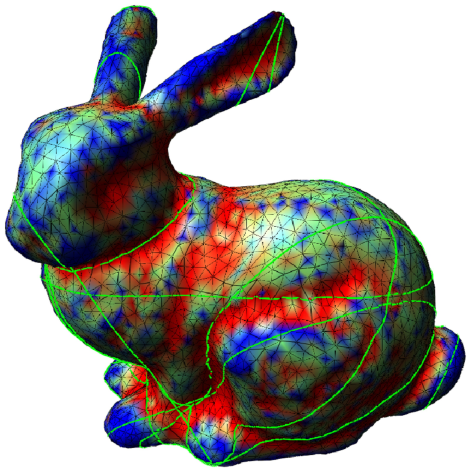
Discrete differential geometry

- Discrete surface: not smooth
- Assumption:

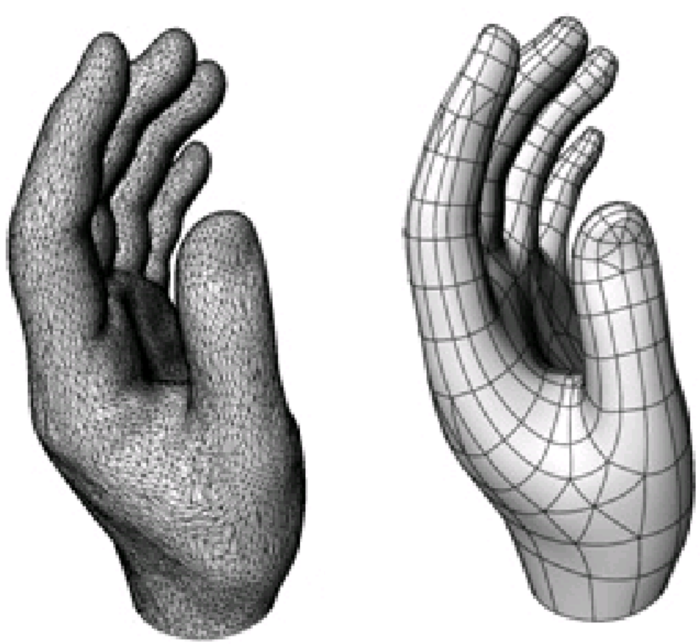
  mesh = piecewise linear approx. of a (real) smooth surface
- Goal:

  find approximations of the differential properties of the underlying smooth surface

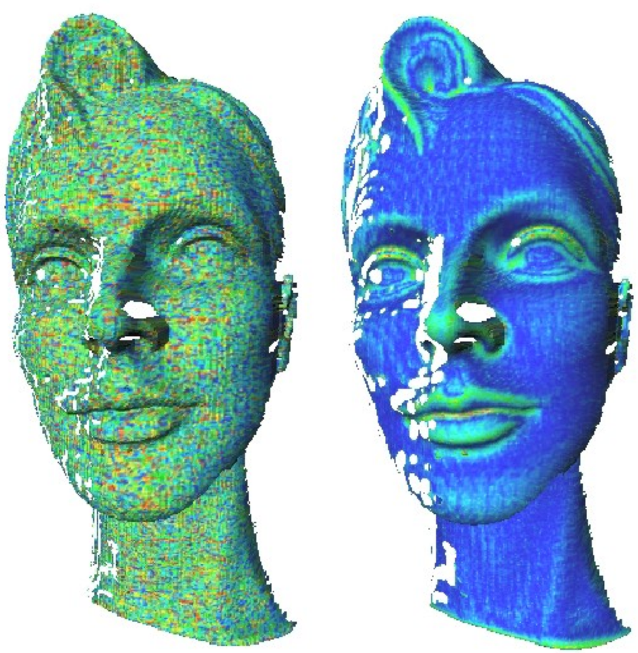
Applications



- Segmentation
- Remeshing
- Denoising or smoothing
- ...



Courtesy P. Alliez



Courtesy M. Meyer

Books

- M. Botsch et al., “Geometric Modeling Based on Polygonal Meshes”, SIGGRAPH 2007 Course Notes, chapters 5 and 6.

http://graphics.ethz.ch/~mbotsch/publications/sg07-course.pdf

http://graphics.ethz.ch/~mbotsch/publications/meshcourse07_code.tgz

- M. Desbrun et al., “Discrete Differential Geometry: An Applied Introduction”, SIGGRAPH 2006 Course Notes

http://ddg.cs.columbia.edu/SIGGRAPH06/DDGCourse2006.pdf

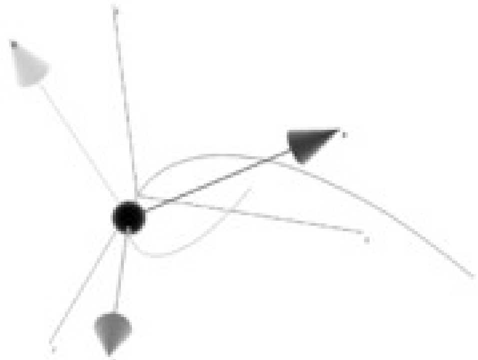
http://ddg.cs.columbia.edu/

Today's planning

1. Differential geometry reminder
2. Discrete curvatures
3. Ridges and ravines
4. Other topics
5. Paper presentations

Differential geometry of a smooth curve

- γ smooth parametric curve: $\gamma : I \to \mathbb{R}^n$
- Frenet vectors/frame:



$$\mathrm{e}_1(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

$$\mathrm{e}_j(t) = \frac{\overline{\mathrm{e}_j}(t)}{\|\overline{\mathrm{e}_j}(t)\|}, \quad \overline{\mathrm{e}_j}(t) = \gamma^{(j)}(t) - \sum_{i=1}^{j-1} \langle \gamma^{(j)}(t), \mathrm{e}_i(t) \rangle \, \mathrm{e}_i(t)$$

- The first 3 vectors are called the tangent, normal and binormal vectors
- If n=3, $\mathrm{e}_3(t) = \mathrm{e}_2(t) \times \mathrm{e}_1(t)$

Curvature of a smooth curve

- Generalized curvature:

$$\chi_i(t) = \frac{\langle \mathbf{e}_i'(t), \mathbf{e}_{i+1}(t) \rangle}{\|\gamma'(t)\|}$$

- Curvature:

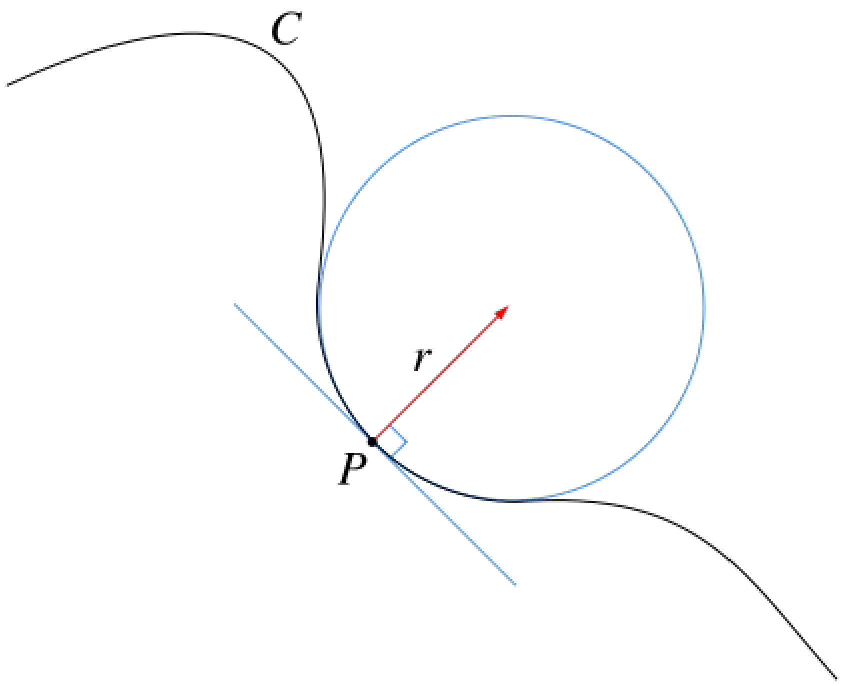
$$\kappa(t) = \chi_1(t) = \frac{\langle \mathbf{e}_1'(t), \mathbf{e}_2(t) \rangle}{\|\gamma'(t)\|}$$

  - Deviance from being a straight line
- Torsion:

$$\tau(t) = \chi_2(t) = \frac{\langle \mathbf{e}_2'(t), \mathbf{e}_3(t) \rangle}{\|\gamma'(t)\|}$$

  - Deviance from being a plane curve

Curvature of a plane curve



- r = curvature radius at P = $1/\kappa(P)$
- Osculating circle
- Exercise: $\gamma(t) = (K.\cos(t), K.\sin(t))$, r = ?

Differential geometry of a smooth surface

- S smooth parametric surface:

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \ (u,v) \in \mathbb{R}^2,$$

- Partial derivatives noted x_u and x_v
- Second partial derivatives noted x_{uu} etc.
- Normal vector: $\mathbf{n} = (\mathbf{x}_u \times \mathbf{x}_v)/\|\mathbf{x}_u \times \mathbf{x}_v\|$

Fundamental forms

- First fundamental form:

$$\mathbf{I} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} := \begin{bmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{bmatrix}$$

- Second fundamental form:

$$\mathbf{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{bmatrix}$$

- Weingarten map/Shape operator:

$$\mathbf{W} := \frac{1}{EG - F^2} \begin{bmatrix} eG - fF & fG - gF \\ fE - eF & gE - fF \end{bmatrix}$$

Curvatures

- Principal directions and principal curvatures:

$$\mathbf{W} = \begin{bmatrix} \bar{\mathbf{t}}_1 & \bar{\mathbf{t}}_2 \end{bmatrix} \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{t}}_1 & \bar{\mathbf{t}}_2 \end{bmatrix}^{-1}$$

- Mean curvature: $H = \dfrac{\kappa_1 + \kappa_2}{2} = \dfrac{1}{2}\mathrm{trace}(\mathbf{W})$
- Gaussian curvature: $K = \kappa_1 \kappa_2 = \det(\mathbf{W})$

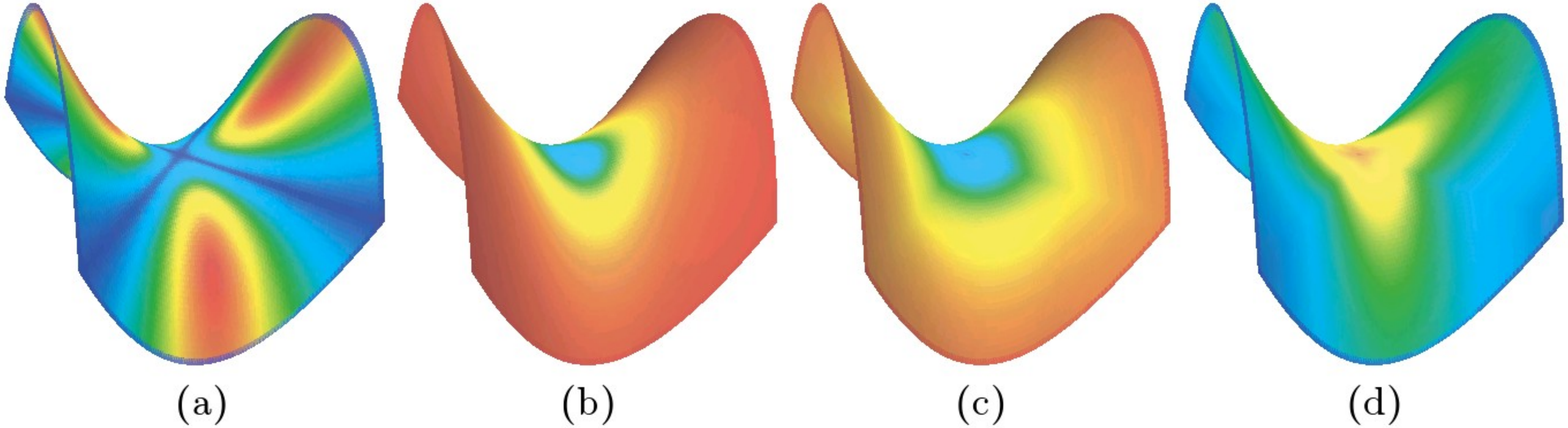
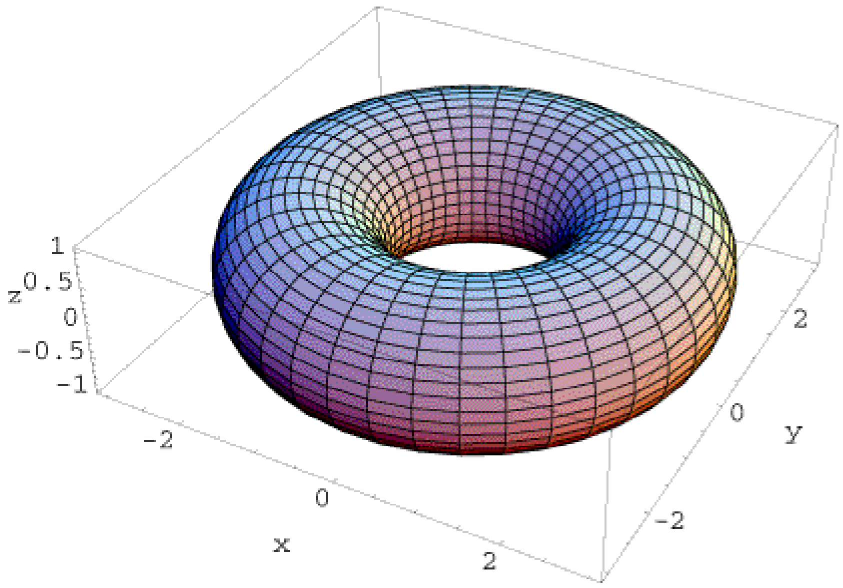


Fig. 5. *Curvature plots of a triangulated saddle using pseudo-colors: (a) Mean, (b) Gaussian, (c) Minimum, (d) Maximum.* (Courtesy M. Meyer)

Example: torus

$$\begin{pmatrix} x[u,v] \\ y[u,v] \\ z[u,v] \end{pmatrix} = \begin{pmatrix} \mathrm{Cos}[u]\,(a + b\,\mathrm{Cos}[v]) \\ (a + b\,\mathrm{Cos}[v])\,\mathrm{Sin}[u] \\ b\,\mathrm{Sin}[v] \end{pmatrix}$$

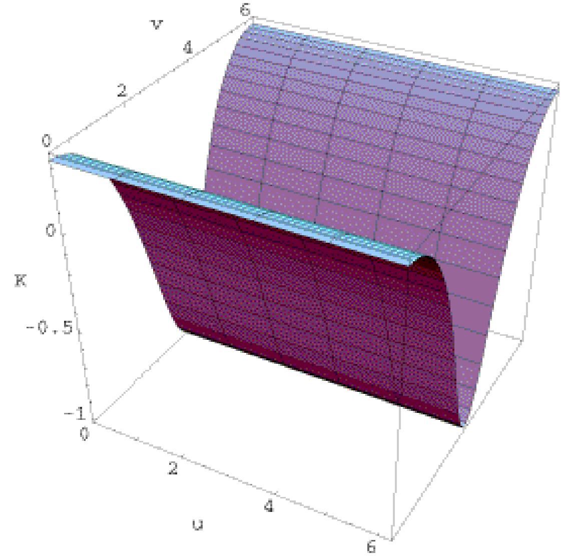


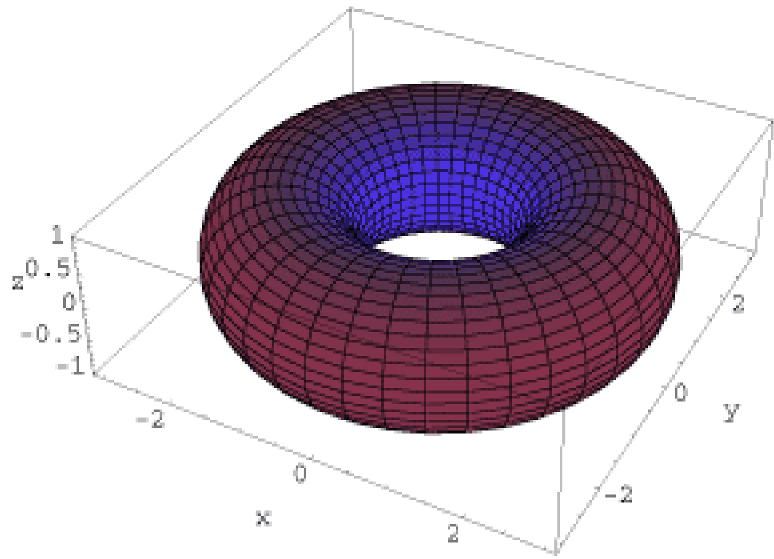
Exercise:

Compute the Gaussian and mean curvatures.

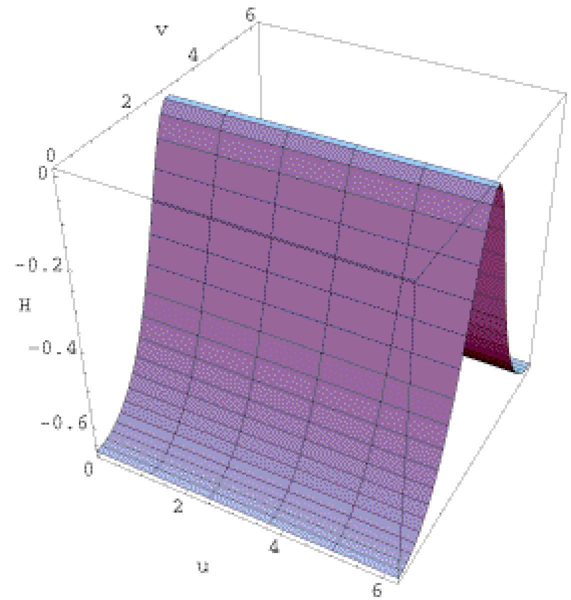
Example: torus

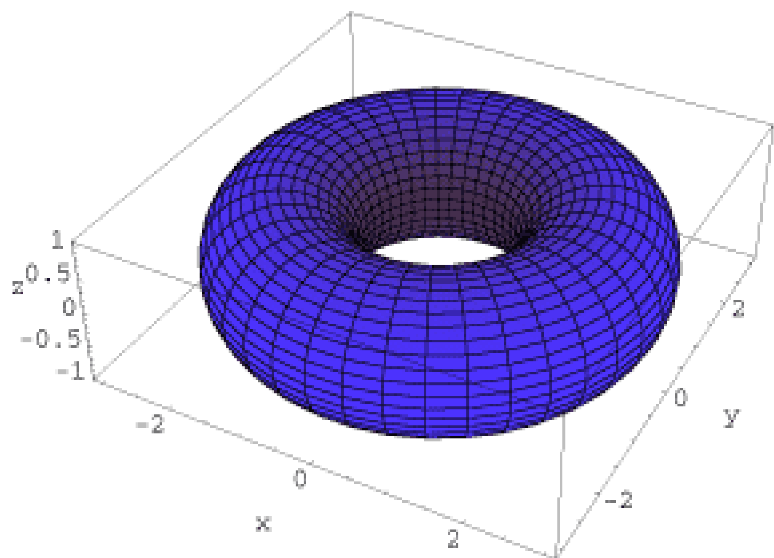
$$K[u, v] = \frac{\cos[v]}{b\,(a + b\cos[v])}$$





$$H[u, v] = -\frac{a + 2\,b\cos[v]}{2\,b\,(a + b\cos[v])}$$





Laplace operator

- Laplace operator in an Euclidean space:

$$\Delta f = \mathrm{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

- Laplace-Beltrami operator for (smooth) manifold surfaces:

$$\Delta_{\mathcal{S}} f = \mathrm{div}_{\mathcal{S}} \nabla_{\mathcal{S}} f$$

- Replacing f by the coordinates function:

$$\Delta_{\mathcal{S}} \mathbf{x} = -2H\mathbf{n}$$

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Discrete differential geometry

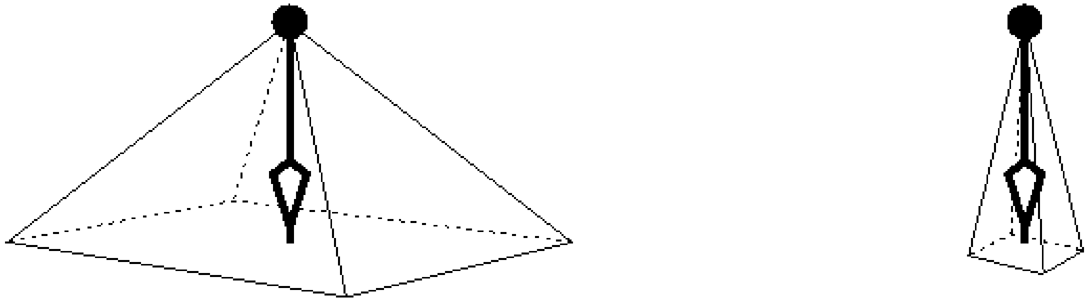
- Discrete surface (here: mesh): not smooth
- Goal: find approximations of the differential properties of the underlying smooth surface
- Idea: express them as averages over a local neighborhood

Discrete Laplace operator

- Taubin 1995:

$$\Delta_{uni} f(v) := \frac{1}{|\mathcal{N}_1(v)|} \sum_{v_i \in \mathcal{N}_1(v)} \left(f(v_i) - f(v) \right)$$

  - local geometry of the discretization (edge lengths, angles) not taken into account
  - bad for non-uniform meshes

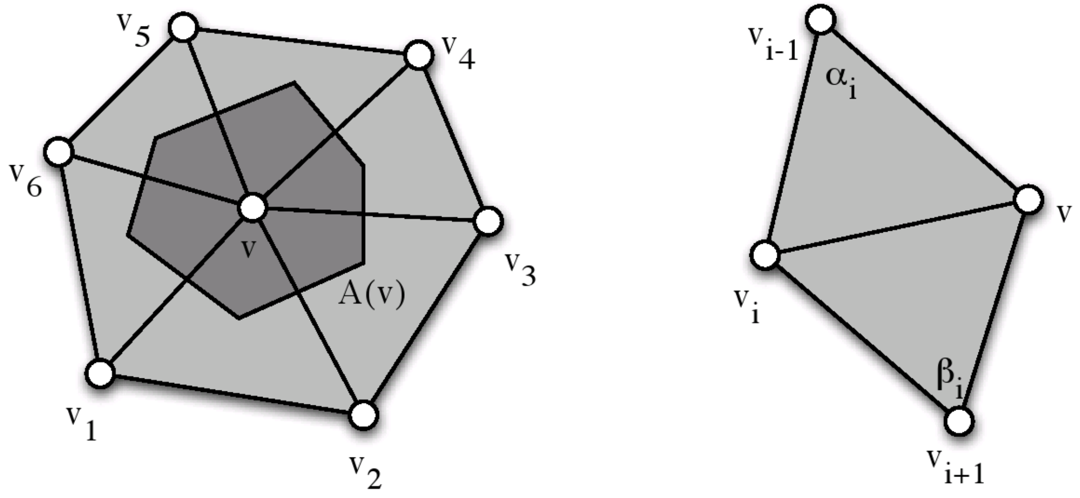


Courtesy M. Desbrun

Discrete Laplace operator

- Pinkall/Polthier 1993, Desbrun et al. 1999:

$$\Delta_{\mathcal{S}} f(v) := \frac{2}{A(v)} \sum_{v_i \in \mathcal{N}_1(v)} (\cot\alpha_i + \cot\beta_i)\,(f(v_i) - f(v))$$



A(v) = Voronoi area

Discrete curvatures

- Mean curvature:

$$\Delta_{\mathcal{S}} \mathbf{x} = -2H\mathbf{n}$$

=> $$H(v) := \frac{1}{A(v)} \sum_{v_i \in \mathcal{N}_1(v)} (\cot\alpha_i + \cot\beta_i) \; || \, v_i - v \, ||$$

- Gaussian curvature:

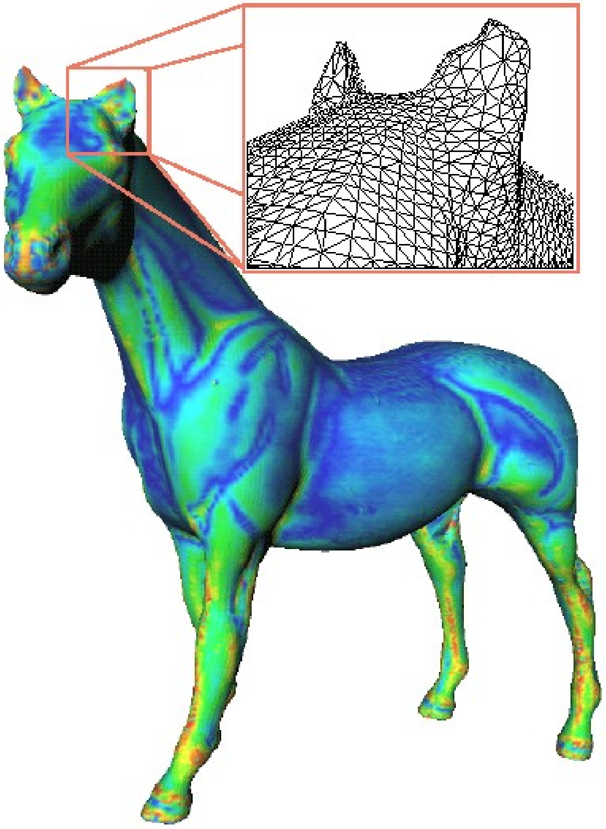
$$K(v) = \frac{1}{A(v)} \left(2\pi - \sum_{v_i \in \mathcal{N}_1(v)} \theta_i \right)$$

Discrete curvatures

Drawback: not intrinsic

- Same surface but ≠ meshes => ≠ curvatures

- Lots of recent work
  - See Grinspun et al. SGP 2007
- My favorite: Cohen-Steiner/Morvan 2003



Courtesy M. Meyer

Cohen-Steiner's definition

$$\mathscr{T}(\mathbf{v}) = \frac{1}{|B|} \sum_{\text{edges } e} \beta(e) \; |e \cap B| \; \; \bar{e}\,\bar{e}^{\,t}$$

curvature tensor

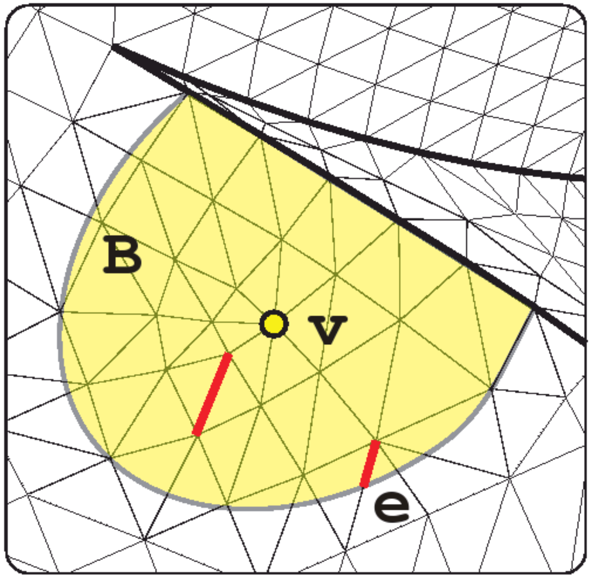
(eigenvectors = normal at v

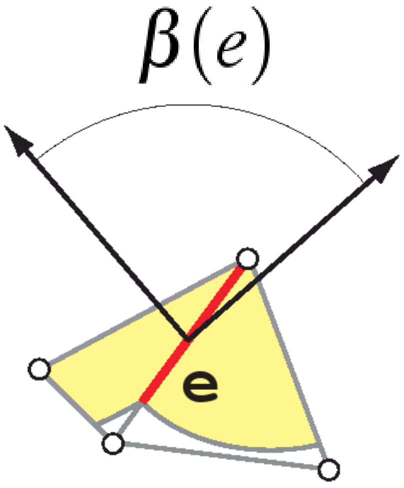
+ principal curvatures)

$\beta(e)$ = **signed** angle

B = approx. geodesic disk

(arbitrary size)

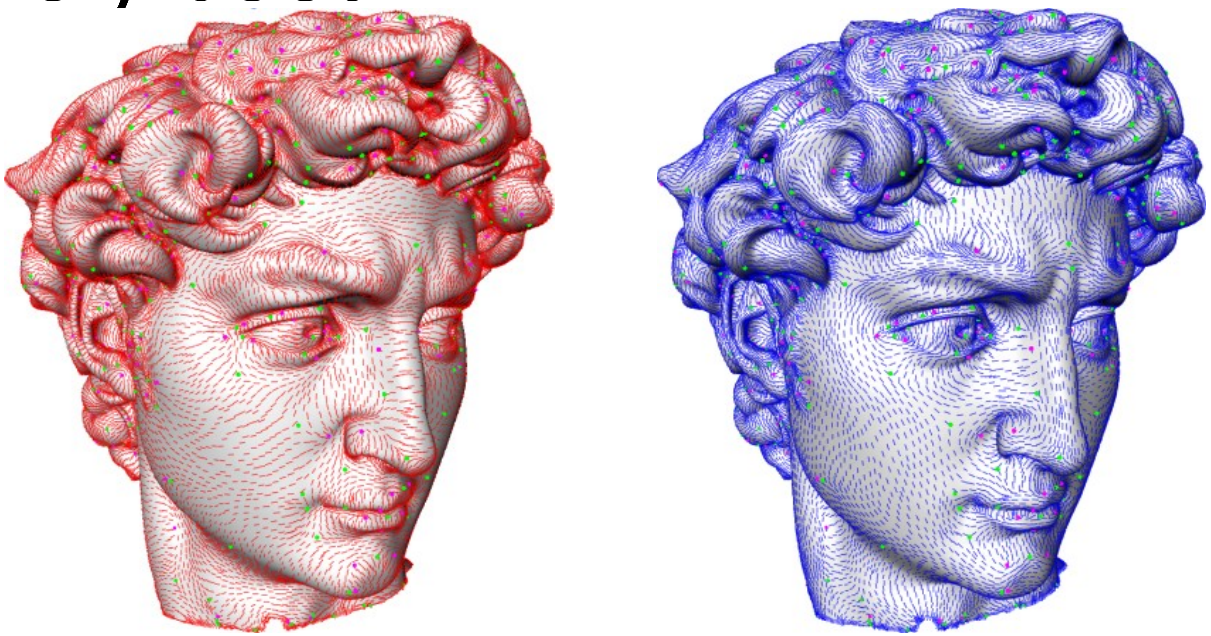




Courtesy P. Alliez

Results

- Based on robust math background
  - Normal cycle theory
- Convergence result w.r.t. smooth surface
- Now widely used



Today's planning

1. ~~Differential geometry reminder~~
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Definition

- Extrema of the principal curvatures along their corresponding curvature direction:

$$e_{\max} = \partial k_{\max} / \partial \mathbf{t}_{\max} \qquad e_{\min} = \partial k_{\min} / \partial \mathbf{t}_{\min}$$

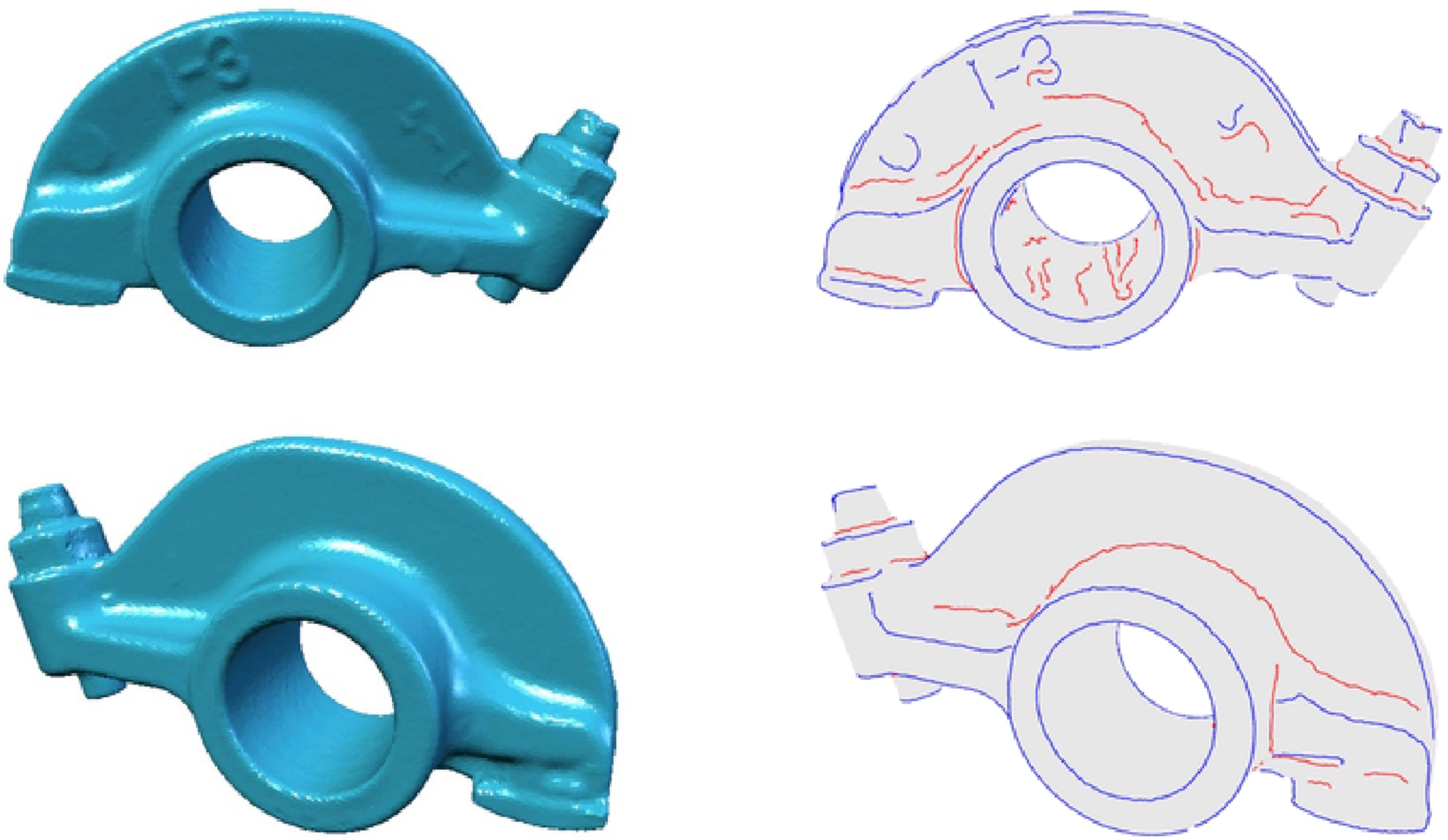
=> $e_{\max} = 0, \quad \partial e_{\max} / \partial \mathbf{t}_{\max} < 0, \quad k_{\max} > |k_{\min}|,$ ridge

$e_{\min} = 0, \quad \partial e_{\min} / \partial \mathbf{t}_{\min} > 0, \quad k_{\min} < -|k_{\max}|$ ravine

- Ridges = (convex) crest lines

Ravines = (concave) crest lines = valleys

Example



Courtesy S. Yoshizawa

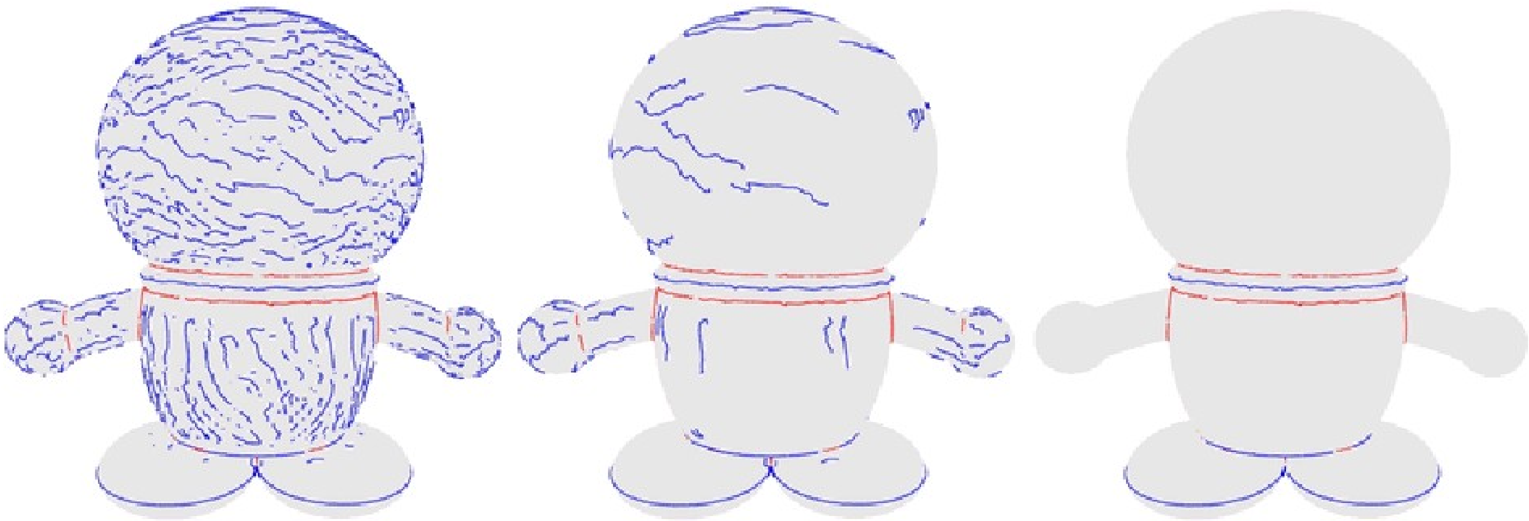
Applications

- Image analysis (medical)
- Face recognition
- Shape analysis
- Compression
- Expressive rendering

Courtesy Y. Ohtake

Computation

- Difficult because:
  - Second order differential quantities
  - Need accurate estimation of principal curvatures
- Most methods need manual filtering



Courtesy S. Yoshizawa

Existing methods

- Use of (discrete) differential operators
  - Noisy
- Polynomial or implicit surface fitting
  - Need less filtering
  - Inherent smoothing difficult to control
  - Slow
  - Local vs. global approaches

Existing methods

- See works by
  - Alexander Belyaev and co-workers
    - Yutaka Ohtake SIGGRAPH 2004
    - Shin Yoshizawa SPM 2005 + Pacific Graphics 2007
  - Frédéric Cazals and Marc Pouget
    - SGP 2003, CAGD 2006
  - Klaus Hildebrandt, Konrad Polthier and Markus Wardetzky
    - SGP 2005

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1. ~~Differential geometry reminder~~
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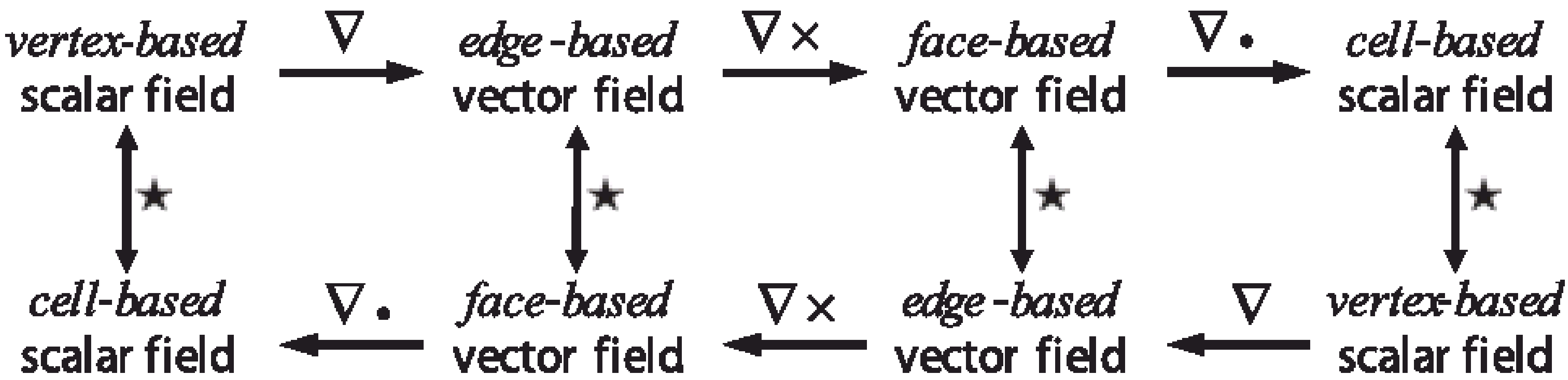
Hot topics in discrete differential geometry

- Still: discrete Laplacian and curvatures
- Exterior Calculus and discrete differential forms
- Curvature-based energies (e.g. Willmore flow) and application to physical simulation
  - Clothes and thin plates
  - Thin shells
- Location of streamlines and singularities
- Harmonic forms

Discrete exterior calculus

- Use of (very) advanced maths to solve various geometrical problems
  - Very general but very abstract
- Keyword: Mathieu Desbrun (Caltech)



Courtesy M. Desbrun

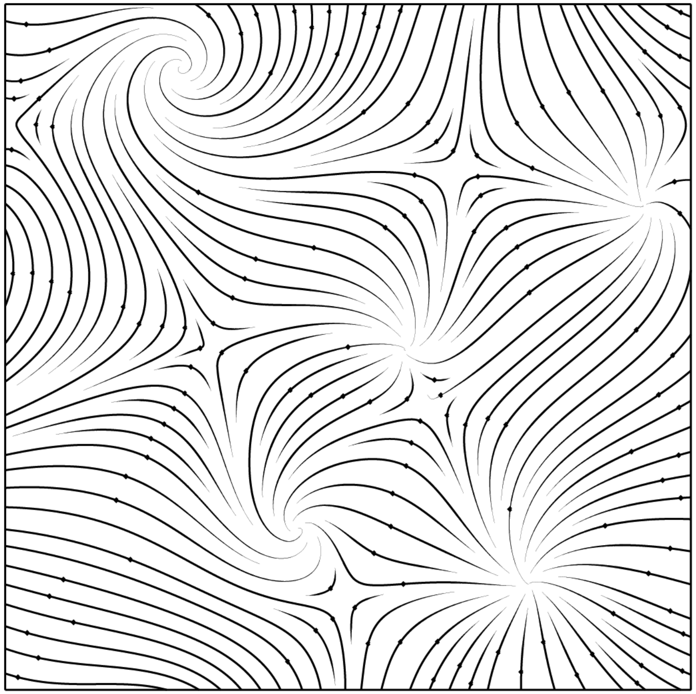
Curvature-based energies

- Needed for (realistic) physical simulation of some special models
  - Clothes
  - Paper sheets
  - Flags
  - ...
- Keyword: Eitan Grinspun (Columbia Univ.)

Courtesy E. Grinspun

Harmonic forms and singularities

- Useful to characterize a shape
  - More info than topology
  - Invariant under some deformations
  - Appl.: parameterization, remeshing
- Keywords:
  - Bruno Lévy (INRIA Lorraine)
  - Pierre Alliez (INRIA Sophia-Antipolis)



Courtesy P. Alliez

The end

- Next week:
  - Mesh parameterization (Cédric Gérot)
  - Point set filtering and simplification (Franck Hétroy)
- These slides will be available on the course's webpage:

http://evasion.imag.fr/Membres/Franck.Hetroy/Teaching/Geo3D/

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1. ~~Differential geometry reminder~~
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Mesh smoothing papers

1. G. Taubin, “A Signal Processing Approach to Fair Surface Design” => F. Hétroy
2. M. Desbrun et al., “Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow”=> P. Bénard & V. Nivoliers
3. S. Fleishman et al., “Bilateral Mesh Denoising”

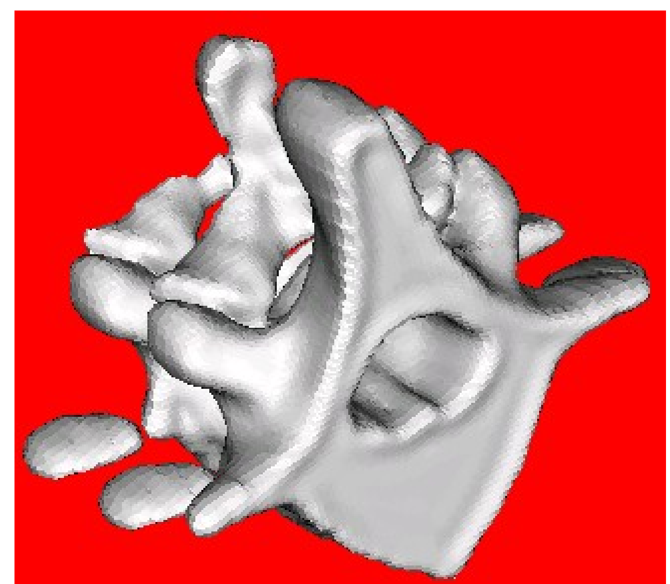
+T.R. Jones et al., “Non-Iterative, Feature-Preserving Mesh Smoothing” => G. Bousquet & J.M. Fernández

Mesh simplification papers

1. H. Hoppe, “Progressive Meshes” => Q. Baig
2. R. Klein et al., “Mesh Reduction with Error Control” => E. Duveau & O. Nagornaya
3. P. Lindström, “Out-of-Core Simplification of Large Polygonal Models”
   => incl. M. Garland & P. Heckbert, “Surface Simplification Using Quadric Error Metrics”
   => A. Méler & V. Vidal

A Signal Processing Approach to Fair Surface Design

- Gabriel Taubin (IBM research)
- Presented at SIGGRAPH 1995
- Fairing = remove rough features (denoise, smooth)
- Main idea: surface fairing ~ signal low-pass filtering



Mathematical approach

- Fourier transform ~ Laplace transform
  - Fourier: $F(t) = cst. \int f(w).\exp(iwt)dw$
  - Laplace: $L(t) = \int f(w).\exp(-wt)dw$
- FT ~ decompose the signal into a linear combination of its Laplacian eigenvectors
- Discrete case: find a equivalent to the Laplacian operator

Discrete Laplacian

- Discrete curve:

$$\Delta x_i = \frac{1}{2}(x_{i-1} - x_i) + \frac{1}{2}(x_{i+1} - x_i)$$

- Discrete surface:

$$\Delta x_i = \sum_{j \in i^\star} w_{ij} \, (x_j - x_i) \quad \text{with} \quad \sum_{j \in i^\star} w_{ij} = 1$$

  - Several choices proposed for the w_{ij}
- Low-pass filtering:

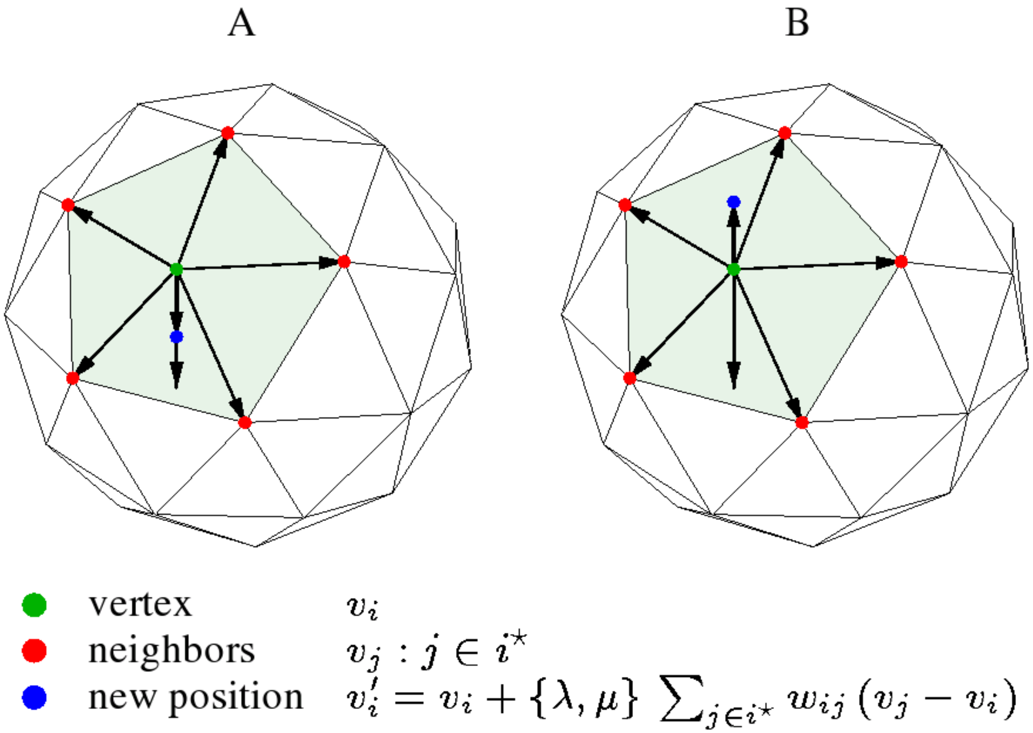
$$x' = f(K)\, x$$

K = Laplacian matrix, f = transfer function

Choice: $f(k) = (1 - \lambda k)(1 - \mu k)$

Fairing algorithm

- 2 steps:
  - $x_i' = x_i + \lambda \, \Delta x_i$ (smoothing)
  - $x_i' = x_i + \mu \, \Delta x_i$ (avoid shrinkage)



Results

- Very fast
  - O(n) (also in memory)
  - FFT: O(n log n)
- OK if the mesh is regular
- Not good if triangles have very different sizes/angles