Mathematics tools 1

December, 15th of 2005 exam

Duration: 2 hours.
All documents are allowed.
The clearness of the answers will be taken into account.
Both parts are independent from each other.

1 Part I: Interpolation and approximation (10 pts)

1.1 Kochanek-Bartels spline

A *Kochanek-Bartels spline* is the concatenation of first order Hermite splines (also called cubic Hermite splines), for which tangents in control points are controlled using three parameters t, b and c varying between -1 and 1.
Let $P_0, \ldots, P_n$ be $n+1$ control points and, $\forall 0 \leq i \leq n-1, C_i$ be a first order Hermite spline defined between P_i and P_{i+1}. We denote D_i and $-S_{i+1}$ the tangents to C_i in its two endpoints (see figure 1).

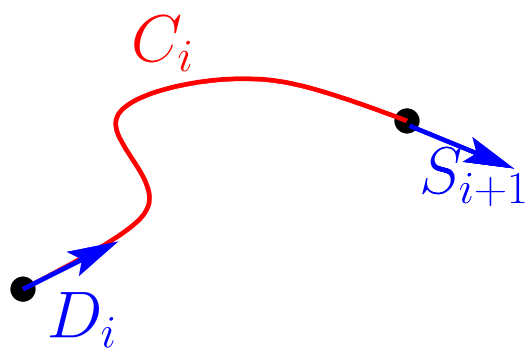


Figure 1: Notation.

The Kochanek-Bartels spline with parameters (t, b, c) for points $P_0, \ldots, P_n$ is defined as the union of splines $C_i, 0 \leq i \leq n-1$, for which $\forall i \in [1, n-1]$

$$S_i = \frac{(1-t)(1+b)(1-c)}{2}(P_i - P_{i-1}) + \frac{(1-t)(1-b)(1+c)}{2}(P_{i+1} - P_i) \tag{1}$$

$$D_i = \frac{(1-t)(1+b)(1+c)}{2}(P_i - P_{i-1}) + \frac{(1-t)(1-b)(1-c)}{2}(P_{i+1} - P_i) \tag{2}$$

1.2 Questions

1. Is a Kochanek-Bartels spline an interpolation spline or an approximation one ?
2. To which known spline corresponds the Kochanek-Bartels spline for which $(t, b, c) = (0, 0, 0)$?
3. For each of the three parameters t, b and c:
   (a) explain its influence on a joining between two Hermite curves C_{i-1} and C_i;
   (b) illustrate this influence drawing sketches for different values of the parameter;
   (c) explain to which "physical" quantity corresponds the parameter.

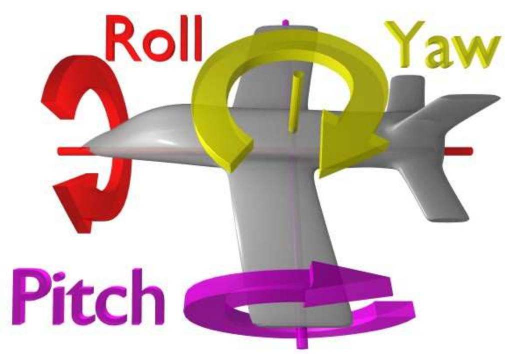
   Advice: you can study the Kochanek-Bartels splines for which
   $(t, b, c) = (1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1)$ and $(0, 0, -1)$.
4. Which other splines do you know, that allow to control more or less the same quantities ?
5. Among interesting properties of splines we studied during the first lecture, which are verified by Kochanek-Bartels splines ?

2 Part II : Position, orientation and motion (10 pts)

2.1 Attitude control of an aircraft

The orientation of an aircraft (or, more generally, of a flying object) in $\mathbb{R}^3$, with respect to a reference frame $\mathcal{R}_0 = (O, \vec{\imath}_0, \vec{\jmath}_0, \vec{k}_0)$, is usually called *attitude*, and is defined by three successive rotations (see figure 2 (a)):

1. a *yaw* rotation (*lacet* in French) around the $\vec{k}_0$ axis;
2. a *pitch* rotation (*tangage* in French) around the axis deduced from $\vec{\jmath}_0$ by previous rotation;
3. a *roll* rotation (*roulis* in French) around the axis deduced from $\vec{\imath}_0$ by the product of the two previous rotations.



(a)

(b)

Figure 2: (a) Aircraft attitude (image from Wikipedia). (b) Roundabout aircraft.

2.2 Questions

1. **Representation using Euler angles.** In order to represent the attitude of an aircraft, the most natural solution is to use Euler angles (also called Cardan angles). Explain how.
2. However, this representation can create a problem, called *gimbal lock* (*verrouillage de cardan* in French), which consists in the loss of a degree of freedom.
   Explain how and give an example leading to a gimbal lock.
3. **Representation using rotation matrices.** In order to avoid this kind of inconvenience, it is adviced to represent attitude another way. Explain how to represent attitude using a matrix. What do you think about this solution ?
4. **Representation using quaternions.** We decide to represent attitude with a quaternion; we denote ϕ, θ and ψ, respectively, the yaw, pitch and roll angles.
   Give the expression of the attitude quaternion with respect to ϕ, ψ and θ. Detail your calculations.
5. What do you think about this solution, compared to the two previous ones ?
6. **Computation of the motion of a roundabout aircraft.** *This question is independent from previous ones.* Eventually, since controlling the attitude of an aircraft seems complex, we choose to restrict ourselves to the case of a roundabout aircraft (see figure 2 (b)). This aircraft is represented by a solid S whose frame is denoted $\mathcal{R}_2$. It goes up and down in the roundabout with a sinusoidal speed: $v_{2,1} = \sin\alpha$. The roundabout, whose frame is denoted $\mathcal{R}_1$, goes round the vertical axis $\overrightarrow{k}_0$ of the world's frame $\mathcal{R}_0$, with constant angular speed ω.
   Compute the motion of the aircraft in the world's frame.