Mathematics tools 1

December, 12th of 2006 exam

Durtion: 2 hours.
All documents are allowed, apart from laptops.
The clearness of the answers will be taken into account.
Both parts are independent from each other.

1 Part I: Interpolation and approximation (10 pts)

1.1 Hierarchical B-spline refinement

We focus on the problem of spline surface *refinement*, that is to say the addition of control points.

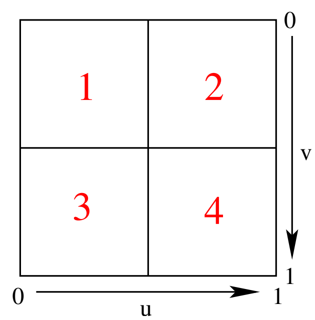

Figure 1: Refinement of a spline surface patch.

1.2 Questions

1. Recall the definition of a uniform (bi)cubic B-spline surface.

2. Let $S(u, v)$ be a uniform bicubic B-spline surface. We note $p_{i,j}$ its $(n+1) \times (m+1)$ control points and $F_i(u)$ or $F_j(v)$ its $n+m+2$ influence functions. Explain why $S(u, v)$ can be piecewise defined, each patch being described in a matricial form:

$$S_{k,l}(u, v) = U\ F_u\ P_{k,l}\ F_v^t\ V^t \tag{1}$$

   with $U = (1\ u\ u^2\ u^3)$, $V = (1\ v\ v^2\ v^3)$ and $P_{k,l}, F_u$ et F_v three matrices to be specified.

3. Suppose that a point $p_{i,j}$ is moved. Which part of the surface $S(u, v)$ is modified ? You can add drawings to your answer to this question and to the following ones if you want.

4. To *refine* $S(u, v)$ means to replace the above $P_{k,l}$ matrix by several matrices $Q_{k,l}$ which can be written as $Q_{k,l} = \alpha_{k,l}\ P_{k,l}\ \beta_{k,l}^t$, with $\alpha_{k,l}$ and $\beta_{k,l}$ matrices made of the coefficients giving the new points w.r.t. the old ones. The simplest example is to divide a surface patch into four patches according to the scheme shown in figure 1. In this case, matrices $\alpha_{k,l}$ and $\beta_{k,l}$ are equal to, respectively, A_1 and A_1 (area 1), A_2 and A_1 (area 2), A_1 and A_2 (area 3), and A_2 and A_2 (area 4), with

$$A_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

   To which notion/concept, seen during lectures, do you think refinement is related ? More precisely, in the case of our example, give the name of one or several famous computer scientist(s).

5. What is the main difference between these two notions ?

6. What is the main interest of refinement if we want to edit and control spline surfaces ? Give an example as accurate as possible.

7. In practice, in the case of *hierarchical* refinements, the new points $q_{i,j}$ are usually described as $q_{i,j} = r_{i,j} + o_{i,j}$, with $r_{i,j}$ the point on the surface maximally influenced by $q_{i,j}$ and $o_{i,j}$ the coordinates of $q_{i,j}$ expressed in a frame associated with the surface and centered in $r_{i,j}$. Do you see an advantage to this modeling ?

2 Part II: Location, orientation and motion (10 pts)

2.1 Error minimization in computer vision

We tackle the problem of pairwise registration of two sets of n points $\mathcal{P}^1 = \{P_i^1\}$ and $\mathcal{P}^2 = \{P_i^2\}$ in $\mathbb{R}^3$, representing the same object seen through two different cameras (see figure 2). Mathematically speaking, we are looking for the best rotation R from $\mathcal{P}^1$ to $\mathcal{P}^2$; this means that we try to minimize the following error:

$$E = \sum_{i=1}^{n} ||P_i^2 - R(P_i^1)||^2 \tag{2}$$

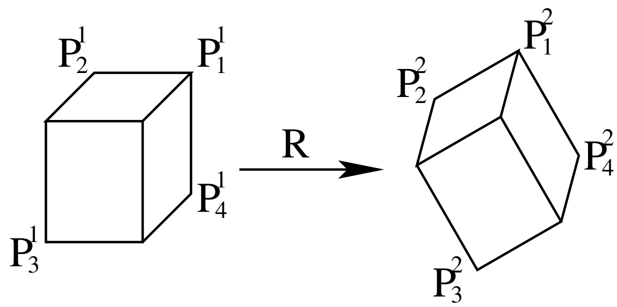

Figure 2: The same object seen through two different cameras.

2.2 Questions

1. In order to compute the optimal rotation R, we must choose how to represent it. What do you think about using:
   (a) matrices ?
   (b) Euler angles ?
   (c) rotation vectors ?
2. We choose to work with quaternions. Let us note q the quaternion which represents the rotation R. Prove that E can be written as:
   $$E = \sum_{i=1}^{n} ||p_i^2 q - q p_i^1||^2 \tag{3}$$
   with p_i^1 and p_i^2 quaternions which must be explained.
3. Prove that, $\forall i \in [1, n], F_i : q \mapsto p_i^2 q - q p_i^1$ is a linear function.
4. Infer from the previous questions that E can be written as:
   $$E = q^t \; A \; q \tag{4}$$
   with A a symmetric matrix (you don't need to compute this matrix).
5. Prove that the minimum of E is given by the unit eigenvector associated to the smallest eigenvalue of A.
6. Conclude (every personal comment about this method is welcomed).
7. Try to formalize a generalization of this problem for n-wise registration; what do you think about this new problem ?