Mathematical tools 1

Session 3

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M2R IVR, October 19th 2006

Interpolation and approximation

Interpolation and approximation

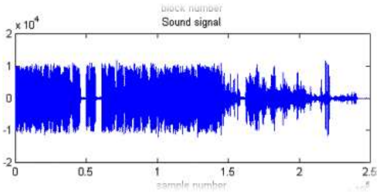
Context

- Process very complex discrete data

$\rightsquigarrow$ huge size/number

$\rightsquigarrow$ complex shapes

Examples



Sound signal

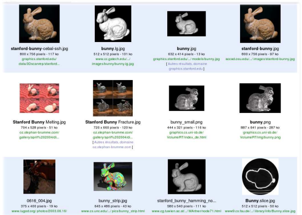
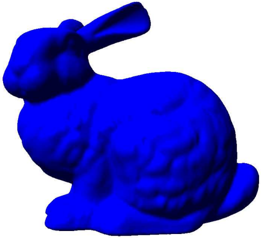


Image database

Medical imaging



3D object

Context

- Process very complex discrete data

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- Analyze these data sometimes globally, sometimes locally

Context

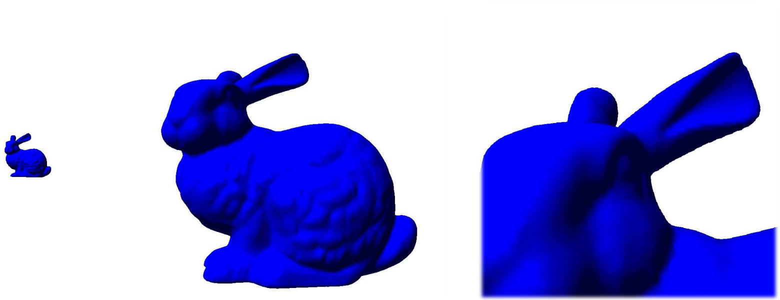
- Process very complex discrete data

  $\rightsquigarrow$ huge size/number

  $\rightsquigarrow$ complex shapes

- Analyze these data sometimes globally, sometimes locally
- Modify these data sometimes globally, sometimes locally

Example



What we need

Several level of details

=

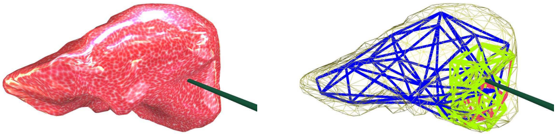
A hierarchical representation of theses complex, discrete data

=

A multi-resolution analysis of the data

Other possible applications

- Data denoising and filtering
- (Medical) image segmentation and feature detection
- Data compression (MPEG-4)
- Querying and matching (images, object reconstruction)
- Real-time animation of deformable objects



from Gilles Debunne's PhD thesis

Interpolation and approximation

Goals

- Decompose a signal into raw signal + details
- Same framework for several signals: 1D, 2D, 2D+t, 3D, ...
- Signal can be approximated at different scales
  $\rightsquigarrow$ *if i coarser than j*, $f_j = f_i + \delta_{i,j}$
- Signal decomposition into and reconstruction from raw + details at any scale can both be done in linear time
- Most details are near zero $\Rightarrow$ efficient compression removing them

Multiresolution analysis

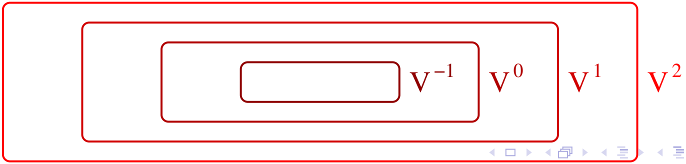
Definition

A multiresolution analysis of a set E is an infinite sequence of nested subsets $(V^j)_{j=-\infty\cdots+\infty}$ such that, among other things:

$$\forall j \in \mathbb{Z}, V^j \subset V^{j+1} \tag{1}$$

$$\lim_{j\to-\infty} V^j = \bigcap_{j=-\infty}^{+\infty} V^j = \{0\} \tag{2}$$

$$\lim_{j\to+\infty} V^j = \overline{\bigcup_{j=-\infty}^{+\infty} V^j} = E \tag{3}$$



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What does that mean ?

(1): signal approx. at high scale is approx at low scale + details
(2): when scale gets coarser and coarser, information gets lost
(3): with approx. at all scales, signal can be fully recovered

Interpolation and approximation

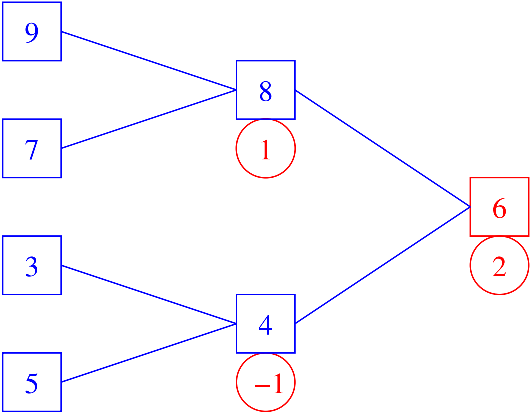
1. Splines

2. Wavelets and multiresolution
   - Context
   - Multiresolution analysis
   - Example: Haar wavelets
   - Subdivision curves and surfaces

1D Haar transform

- Input: signal = sequence of $N = 2^n$ values $(u_i)_{0 \leq i \leq N-1}$
- What we want: to decompose the signal into one mean value + details
- How to do that:
  - Recursively replace u_{2k} and u_{2k+1} by $\frac{u_{2k}+u_{2k+1}}{2}$
  - Keep as detail $u_{2k} - \frac{u_{2k}+u_{2k+1}}{2} = \frac{u_{2k}-u_{2k+1}}{2}$

Example



Discussion

- Def.: Wavelet transform of the signal = mean value followed by details in order of increasing resolution
- Decomposition and reconstruction can both be done in $O(n)$ time
- Mean value + all details = exactly n terms stored in memory
- No information is lost or created during the process
- Application to compression: remove details ~ 0

1D Haar basis functions

- From the discrete to the continuous space:
  now signal = piecewise-constant function on $[0, 1[$
- V^0 = vector space of constant functions on $[0, 1[$
- V^1 = vector space of constant functions on $[0, \frac{1}{2}[$ and on $[\frac{1}{2}, 1[$
- V^2 = vector space of constant functions on $[0, \frac{1}{4}[$, on $[\frac{1}{4}, \frac{1}{2}[$, on $[\frac{1}{2}, \frac{3}{4}[$ and on $[\frac{3}{4}, 1[$
- etc.
- (V^j) are nested vector spaces: $V^0 \subset V^1 \subset V^2 \subset \dots$

1D Haar basis functions

- Basis functions of each V^j: box functions

$$\phi_i^j(x) = \phi(2^j x - i), i = 0 \ldots 2^j - 1$$

with

$$\phi(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- These functions are called scale factors

Exercise

Example

Draw the basis functions of V^2.

Exercise

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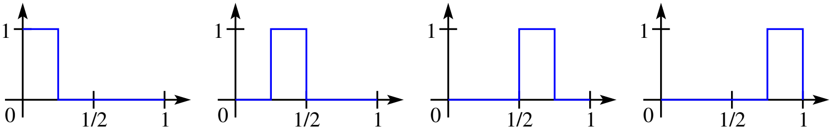
Draw the basis functions of V^2.

Solution:



Orthogonality

Definition (Inner product)

For $f, g : V^j \to \mathbb{R}, < f|g >= \int_0^1 f(x)g(x)dx$.

Definition (Orthogonality)

f and g are orthogonal if $< f|g >= 0$.

Property

$\forall j$, the basis $(\phi_i^j)_{i=0\ldots 2^j-1}$ of V^j is orthogonal (i.e. the ϕ_i^j are orthogonal in pairs).

Wavelets

Definition (Orthogonal supplement of V^j in V^{j+1})

W^j = space of all functions in V^{j+1} which are orthogonal to all functions in V^j. We write $V^j \oplus W^j = V^{j+1}$.

Definition

Basis functions ψ_i^j of W^j are called wavelets.

Property

$(\phi_i^j)_{i=0\dots 2^j-1} \cup (\psi_i^j)_{i=0\dots 2^j-1}$ *is a basis of* V^{j+1}.

Exercise

Example

Find Haar wavelets for W^1.

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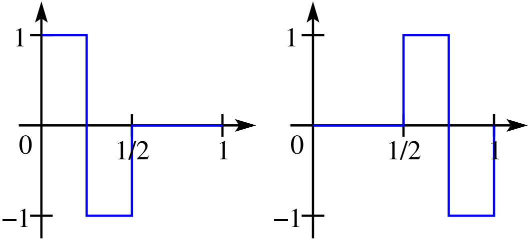
Find Haar wavelets for W^1.

Solution:



Exercise

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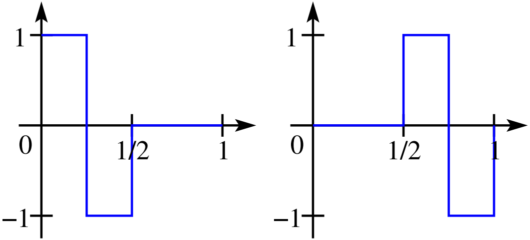
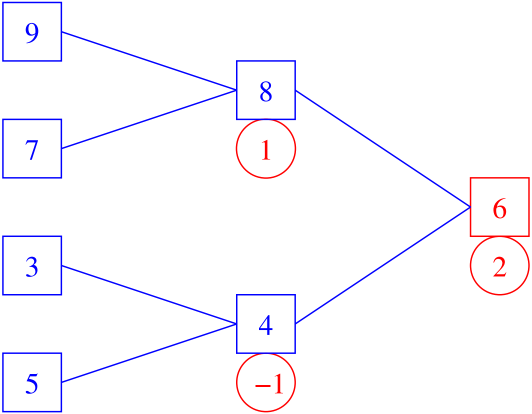
Solution:



$$\psi(x) = \begin{cases} 1 & \text{for } 0 \le x < \frac{1}{2} \\ -1 & \text{for } \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Back to first example



Haar basis

Property

Since $V^j \oplus W^j = V^{j+1}$, *we have*

$$V^{j+1} = V^0 \oplus W^0 \oplus W^1 \oplus \cdots \oplus W^j$$

Haar basis

Property

Since $V^j \oplus W^j = V^{j+1}$, we have

$$V^{j+1} = V^0 \oplus W^0 \oplus W^1 \oplus \cdots \oplus W^j$$

Definition (Haar basis)

The Haar basis of V^{j+1} includes ϕ_0^0 (basis function of V^0), ψ_0^0 (basis function of W^0), ψ_0^1, ψ_1^1 (basis functions of W^1), ..., $\psi_0^j, \ldots, \psi_{2^j-1}^j$ (basis functions of W^j).

Haar basis

Property

Since $V^j \oplus W^j = V^{j+1}$, we have

$$V^{j+1} = V^0 \oplus W^0 \oplus W^1 \oplus \cdots \oplus W^j$$

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The Haar basis of V^{j+1} includes ϕ_0^0 (basis function of V^0), ψ_0^0 (basis function of W^0), ψ_0^1, ψ_1^1 (basis functions of W^1), ..., $\psi_0^j, \ldots, \psi_{2^j-1}^j$ (basis functions of W^j).

$$\phi_i^j(x) = \phi(2^j x - i)$$

$$\psi_i^j(x) = \psi(2^j x - i)$$

Orthonormal Haar basis

Property

Any Haar basis is orthogonal.

Property

A Haar basis can be normalized (i.e., $\forall u$ in this basis, $< u|u >= 1$) if we replace previous definitions with

$$\phi_i^j(x) = \sqrt{2^j}\phi(2^j x - i)$$

$$\psi_i^j(x) = \sqrt{2^j}\psi(2^j x - i)$$

Wavelet decomposition

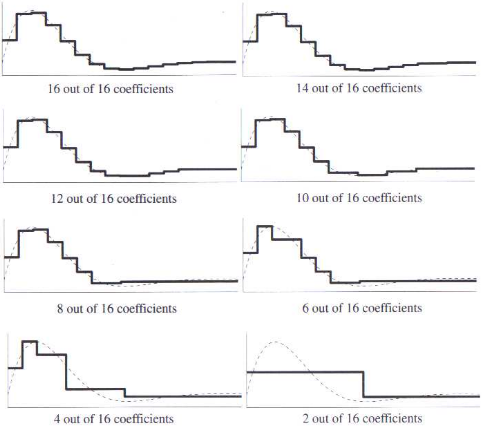
```
procedure Decomposition(c: array[1..2^j] of reals)
    c := c/√(2^j); // normalize input coefficients
    g := 2^j; // size of data to decompose
    while g ≥ 2 do
        DecompositionStep(c[1..g]);
        g := g/2;
    end while;
end Decomposition;

procedure DecompositionStep(c: array[1..2^j] of reals)
    for i := 1 to 2^(j−1) do
        c'[i] := (c[2*i-1]+c[2*i])/√2; // mean value
        c'[2^(j−1)+i] := (c[2*i-1]-c[2*i])/√2; // detail
    end for;
    c := c';
end DecompositionStep;
```

Reconstruction

procedure Reconstruction(c: **array**[1..2^j] **of reals**)
    g := 2; // size of data to reconstruct
    while $g \leq 2^j$ **do**
        ReconstructionStep(c[1..g]);
        g := 2*g;
    end while;
    c := c*$\sqrt{2^j}$; // undo normalization
end Reconstruction;

procedure ReconstructionStep(c: **array**[1..2^j] **of reals**)
    for i := 1 to 2^{j-1} **do**
        c'[2*i-1] := (c[i]+c[2^{j-1}+i])/$\sqrt{2}$;
        c'[2*i] := (c[i]-c[2^{j-1}+i])/$\sqrt{2}$;
    end for;
    c := c';
end ReconstructionStep;

Wavelet compression



Algorithm: sort coefficients in decreasing order, remove lasts

Haar wavelets for images

- 2 ways to decompose an image:
  1. Standard decomposition: first decompose rows, then columns
  2. Nonstandard decomposition: alternate between row and column decomposition
- Also for reconstruction
- Advantages:
  - Standard: only 1D transforms are performed
  - Nonstandard: more efficient ($\frac{8}{3}(n^2 - 1)$ assignment operations instead of $4(n^2 - n)$).

Standard decomposition

procedure StandardDecomposition(c: **array**[1..2^j,1..2^k] **of reals**)
    for row := 1 to 2^j **do**
        Decomposition(c[row,1..2^k]);
    end for;
    for col := 1 to 2^k **do**
        Decomposition(c[1..2^j,col]);
    end for;
end StandardDecomposition;

procedure StandardReconstruction(c: **array**[1..2^j,1..2^k] **of reals**)
    for col:= 1 to 2^k **do**
        Reconstruction(c[1..2^j,col]);
    end for;
    for row := 1 to 2^j **do**
        Reconstruction(c[row,1..2^k]);
    end for;
end StandardReconstruction;

Standard decomposition

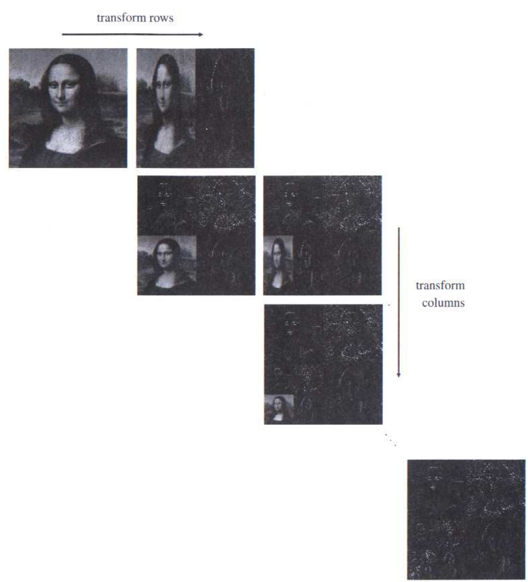
Nonstandard decomposition

procedure NonstandardDecomposition(c: **array**[1..2^j,1..2^k] **of reals**)
    c := c/2^j; // Normalize input coefficients
    g := 2^j;
    while $g \geq 2$ **do**
        for row := 1 to g **do**
            DecompositionStep(c[row,1..g]);
        end for;
        for col := 1 to g **do**
            DecompositionStep(c[1..g,col]);
        end for;
        g := g/2;
    end while;
end NonstandardDecomposition;

procedure NonstandardReconstruction(c: **array**[1..2^j,1..2^k] **of reals**)
    //*** TO BE COMPLETED ***

end NonstandardReconstruction;

Nonstandard decomposition



Book

All stuff in this section (Haar wavelets) has been inspired by the first chapter of the following book:

E. Stollnitz, T. DeRose, D. Salesin
Wavelets for Computer Graphics
Morgan Kaufmann, 1996

See you next week

The end !

Interpolation and approximation