Creating & processing 3D geometry:
smooth boundary representations

1. Representations
   - Discrete models: points, meshes, voxels
   - Smooth boundary: Parametric & Subdivision surfaces
   - Smooth volume: Implicit surfaces

2. Geometry processing
   - Smoothing, simplification, parameterization

3. Creating geometry
   - Reconstruction
   - Interactive modeling, sculpting, sketching

Why do we need Smooth Surfaces?

Meshes
   - Explicit enumeration of faces
   - Many required to be smooth!
   - Smooth deformation???

Smooth surfaces
   - Compact representation
   - Will remain smooth
     - After zooming
     - After any deformation!

Parametric curves and surfaces

Defined by a parametric equation
   - Curve: \( C(u) \)
   - Surface: \( S(u,v) \)

Advantages
   - Easy to compute point
   - Easy to discretize
   - Parametrization

Parametric curves: Splines

Motivations: interpolate/approximate points \( P_k \)
   - Easier to give a finite number of “control points”
   - The curve should be smooth in between

Why not polynomials? Which degree do we need?

Spline curves

- Defined from control point
- Local control
  - Joints between polynomial curve segments
  - Degree 3, \( C^2 \) or \( C^1 \) continuity

Choice of a representation?

Notion of “geometric model”
   - Mathematical description of a virtual object
     (enumeration/equation of its surface/volume)

How should we represent this object…
   - To get something smooth where needed?
   - To have some real-time display?
   - To save memory?
   - To ease subsequent deformations?
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Interpolation vs. Approximation

Splines curves

Mathematical formulation?
- Curve points \( = \) linear combination of control points

Desirable properties for the "influence functions" \( F_k \)?

Properties of influence functions?

1. Affine invariance

Invariance to affine transformations?
- Same shape if control points are translated, rotated, scaled

\[ \sum F_k(u) = 1 \]

- Influence coefficients are barycentric coordinates
- Prop: barycentric invariance too. Application to morphing

Properties of influence functions?

2. Convex hull

Convex hull: \( F_k(u) \geq 0 \)

- Curve points are barycenters

- Draw a normal, positive curve which interpolates
- Can it be smooth?

Properties of influence functions?

3. Variance reduction

No unwanted oscillation?

No intersections curve / plane \( \leq \) control polygon / plane

- A single maximum for each influence function

Properties of influence functions?

4. Locality

Local control on the curve?
- easier modeling, avoids re-computation

Choose \( F_k \) with local support
- Zero and zero derivatives outside an influence region
- Are they really polynomials?
Properties of influence functions?
5. Continuity: parametric / geometric

- Parametric continuity C1, C2, etc
  - Easy to check
  - Important if the curve defines a trajectory!
  Ex: \( q(u) = (2u, u) \), \( r(t) = (4t+2, 2t+1) \).
  Continuity at \( J = q(1) = r(0) \) ?

- Geometric continuity G1, G2, etc

Splines curves
Summary of desirable properties

\[ C(t) = \sum F_k(t) P_k \]

- Interpolation & approximation
  - Affine invariance \( \sum F_k(t) = 1 \)
  - Locality \( F_k(t) \) with compact support
  - Parametric or geometric continuity

- Approximation
  - Convex envelope \( F_k(t) \geq 0 \)
  - Variance reduction: no unwanted oscillation

Splines curves
Most important models

- Interpolation
  - Hermite curves C1, cannot be local if C2
  - Cardinal spline (Catmull Rom)

- Approximation
  - Bézier curves
  - Uniform, cubic B-spline (unique definition, subdivision)
  - Generalization to NURBS

Cardinal Spline, with tension=0.5

Uniform, cubic Bspline

Cubic splines: matrix equation

\[ Q_i(u) = (u^3 \ u^2 \ u \ 1) \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & -3 & 0 & 1 \end{bmatrix} P_i \]

\[ M_{spline} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & -3 & 0 & 1 \end{bmatrix} \]

\[ M_{knot} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & -3 & 0 & 1 \end{bmatrix} \]
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Splines surfaces

- Tensor product: product of spline curves in \( u \) and \( v \)
  \[ Q_{i,j}(u, v) = (u^3 u^2 u 1) \cdot (v^3 v^2 v 1) \]

- Smooth surface?
- Convert to meshes?
- Locality?

Splines surfaces

- Expression with separable influence functions!
  \[ Q_{i,j}(u, v) = \sum B_i(u) B_j(v) P_{ij} \]

Historic example

Can splines represent complex shapes?

- Fitting 2 surfaces: same number of control points

Can splines represent Complex Shapes?

Closed surfaces can be modeled
- Generalized cylinder: duplicate rows of control points
- Closed extremity: degenerate surface!

Can we fit surfaces arbitrarily?

Subdivision Curves & Surfaces

- Start with a control polygon or mesh
- Progressive refinement rule (similar to B-spline)
- Smooth? use variance reduction!
  - "corner cutting"
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**How Chaikin’s algorithm works?**

\[
Q_i = \frac{3}{4} P_i + \frac{1}{4} P_{i+1} \\
R_i = \frac{1}{4} P_i + \frac{3}{4} P_{i+1}
\]

**Subdivision Surfaces**

- Topology defined by the control polygon
- Progressive refinement (interpolation or approximation)

**Example: Butterfly Subdivision Surface**

- Interpolate
- Triangular
- Uniform & Stationary
- Vertex insertion (primal)
- 8-point

\[ a = \frac{1}{2}, \quad b = \frac{1}{8} + 2w, \quad c = -\frac{1}{16} - w \]

\[ w \text{ is a tension parameter} \]

\[ w = -\frac{1}{16} \Rightarrow \text{surface isn’t smooth} \]

**Example: Doo-Sabin**

Works on quadrangles; Approximates

**Comparison**

<table>
<thead>
<tr>
<th>Catmull-Clark (primal)</th>
<th>Doo-Sabin (dual)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Original control mesh.</td>
<td>(b) Control mesh after one subdivision.</td>
</tr>
</tbody>
</table>
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**Subdivision Surfaces**

**Benefits**
- Arbitrary topology & geometry (branching)
- Approximation at several levels of detail (LODs)

**Drawback:** No parameterization, some unexpected results

Extension to multi-resolution surfaces: Based on wavelets theory

**Advanced bibliography**

1. **Generalized B-spline Surfaces of Arbitrary Topology**
   [Charles Loop & Tony DeRose, SIGGRAPH 1990]
   - n-sided generalization of Bézier surfaces: “Spatches”

2. **Xsplines** [Blanc, Schlick SIGGRAPH 1995]
   Approximation & interpolation in the same model

3. **Exact Evaluation of Catmull-Clark Subdivision**
   [Jos Stam, Siggraph 98]
   Analytic evaluation of surface points and derivatives
   - Even near irregular vertices,
   - At arbitrary parameter values!
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Advanced bibliography

4. Subdivision Surfaces in Character Animation

[Tony DeRose, Michael Kass, Tien Truong, Siggraph 98]

Keeping some sharp creases where needed

Advanced bibliography

5. T-splines & T-NURCCs [Sederberg et al., Siggraph 2003]

T-splines d³, C²: superset of NURBS, enable T junctions!
- Local lines of control points
- Eases merging

T-NURCCs: Non-Uniform Rational Catmull-Clark Surfaces with T-junctions
- Superset of T-splines & Catmull-Clark
- Enable local refinement
- Same limit surface.
- C2 except at extraordinary points.

Comment représenter la géométrie ?

• Représentations par bord / surfaciques / paramétriques
  - Polygones (surfaces discrètes)
  - Surfaces splines
  - Surfaces de subdivision, surfaces multi-résolution
• Représentations volumiques / implicites
  - Voxels (volumes discrets)
  - CSG (Constructive Solid Geometry)
  - Surfaces implicites

Adapter le choix aux besoins de l’animation et du rendu!